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A Vector Spline Approximation

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We introduce a new family $P_{\alpha,\beta}$ of spline minimization problems for vector fields, defined by

$$P_{\alpha,\beta}\begin{cases}\text{Min}\left(\alpha\int_{\mathbb{R}^2}\|\nabla\,\text{div}\,V\|^2\,dx\,dy+\beta\int_{\mathbb{R}^2}\|\nabla\,\text{rot}\,V\|^2\,dx\,dy\right)\\ V\in\mathscr{X}\quad\text{and}\quad V(X_i)=V_i,\qquad i=1,\ldots,N,\end{cases}$$

where $V=(u,v)$ is a two component vector function, $\mathscr{X}$ is the Beppo Levi space $D^{-2}L^2(\mathbb{R}^2)\times D^{-2}L^2(\mathbb{R}^2)$, $X_i=(x_i,y_i)$ are the interpolation points, and $V_i=(u_i,v_i)$ are data values. A coupling between V components is achieved by the divergence (div) and rotational (rot) operators. α, β are fixed real positive constants controlling the relative weight on the gradient of the divergence and rotational fields. The explicit control on divergence and rotational operators is well suited for geophysical fluid flow interpolations; it allows us to cope with the great differences frequently observed in the magnitudes of the divergent and rotational parts of the flow. Through the general spline formalism, existence and uniqueness of the solution is proved. The analytical solution is explicitly calculated and numerical examples are presented. For α (and β) $\to 0$, "limit" problems are defined and their analytical solutions are given. © 1991 Academic Press, Inc.

1. INTRODUCTION

In meteorology, interpolation methods are essential for the restitution of physical fields from observed data, generally irregularly located in space and time. The fields, produced by meteorological analysis, are used as

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initial value fields for numerical weather prediction models and for diagnostic purposes.

In horizontal wind field analysis, derived divergence (div $V = \partial_x u + \partial_y v$) and vorticity (also called rotational; rot $V = \partial_x v - \partial_y u$) fields are of prime importance. For example, the horizontal divergence field allows the diagnosis of vertical velocity (not measured by operational meteorological observing systems); also, potential–kinetic energy conversion is controlled by the divergence field.

In this context, vector field interpolation introduces some specific problems. The simplest approach is to treat the two components separately. The result, as pointed out by many authors (cf. Daley [4, 5], Pedder [15]), shows that the lack of intercomponent correlation often generates unrealistic fluctuations in the derived divergence and rotational fields. This observation suggests the use of a smoothing operator applied on these fields; in this way, V components are intrinsically linked. Moreover, an a prioric scale analysis of the equations of motion reveals that the divergence and rotational fields are generally not of the same order of magnitude: for example, in synoptic midlatitude processes, we have $|\text{div } V|/|\text{rot } V| \simeq \text{Ro} \simeq 0.1$ (Ro is the Rossby number, $\text{Ro} = U/fL$; U is the velocity scale, $U \simeq 10\ m/s$; f is the Coriolis parameter, $f \simeq 10^{-4}\ s^{-1}$; L is the length scale, $L \simeq 10^6 m$; cf. Haltiner and Williams [11], Chap. 3). In fact, a divergence–rotational control is intended to fit this information.

At present, statistical regression methods (i.e., minimum variance linear estimates, called optimal interpolation: cf. Eliassen [9], Gandin [10], Daley [4, 5]) are currently used in meteorological analysis. For wind interpolations, multivariate formulations are used, in which intercomponent correlations are explicitly considered; the control on divergence–rotational fields is achieved by a statistical hypothesis on the correlation functions modelling the velocity potential χ and the stream function ψ associated to V ($V = \nabla\chi + \text{Rot } \psi$; cf. Section 7.2).

The solution we present here is formulated in the variational spline formalism and extends, in some aspects, the thin-plate spline introduced by Duchon (cf. [8]). It gives a spline functional equivalence to the statistical approach. A similar approach to vector interpolation, showing an intercomponent coupling, has recently been formulated by Atteia and Benbourhim (cf. [1]).

In the next section, we present the minimization problem leading to the vector spline. The solution of this problem depends on two positive parameters, α and β. The particular case of uncoupled (u, v) is obtained for $\alpha = \beta$, for which the problem splits into a couple of thin-plate spline problems, one for each of the components.

In the third section, existence and uniqueness of the solution is established.

The fourth section gives the analytical expression of the vector spline.

The fifth section discusses the so-called limit problems, obtained for $\alpha/\beta \to 0$ (or ∞). A natural decomposition of the Hilbert space $\mathscr{X}$ follows.

Some numerical examples are presented in the sixth section.

In the last section, we briefly discuss the problem of optimal values of the parameters α and β; the statistical approach leading to the same vector leading to the same vector interpolation is outlined in terms of vector Kriging.

2. The Minimization Problem

Let $D^{-2}L^2(\mathbb{R}^2) = \{u \in \mathscr{D}'(\mathbb{R}^2)/D^\gamma u \in L^2(\mathbb{R}^2), |\gamma| = 2\}$, the Beppo–Levi space of order 2; $\mathscr{D}'(\mathbb{R}^2)$ is the space of distributions in $\mathbb{R}^2$ and $\mathscr{D}(\mathbb{R}^2)$ (or $\mathscr{D}$) the test function space.

In the Hilbert space $D^{-2}L^2$, we consider the scalar product $(u \mid u') = \sum_{i=1}^3 u(X_i)u'(X_i) + ((u \mid u'))$, where $((u \mid u'))$ is the semi-scalar product defined by $((u \mid u')) = \sum_{|\gamma|=2} \int_{\mathbb{R}^2} D^\gamma u D^\gamma u' \, dx\, dy$, $u, u' \in D^{-2}L^2$, and $X_i = (x_i, y_i)$, $i = 1, 2, 3$, are three non-aligned points. We denote by $\mathscr{P}_1$ the set of polynomials defined on $\mathbb{R}^2$ of total degree less than or equal to 1.

A fundamental result on Beppo–Levi space is given by

PROPOSITION 2.1. *$\mathscr{D} + \mathscr{P}_1$ is dense in $D^{-2}L^2$.*

Proof. Consider $u \in D^{-2}L^2$ orthogonal to the space $\mathscr{D} + \mathscr{P}_1$; we have

$$(u \mid u') = \sum_{i=1}^{3} u(X_i)u'(X_i) + \sum_{|\gamma|=2} \int_{\mathbb{R}^2} D^\gamma u D^\gamma u' \, dx\, dy = 0, \tag{2.1}$$

$\forall u' \in \mathscr{D} + \mathscr{P}_1$. If we take $u_i' \in \mathscr{P}_1$ verifying $u_i'(X_j) = \delta_{ij}$, $i, j = 1, 2, 3$ (δ_{ij} is the Kronecker symbol), Eq. (2.1) shows that $u(X_i) = 0$. Consider then $u' = \varphi \in \mathscr{D}$; Eq. (2.1) gives $\sum_{|\gamma|=2} \int_{\mathbb{R}^2} D^\gamma u D^\gamma \varphi \, dx\, dy = 0$ and $\int_{\mathbb{R}^2} \Delta^2 u \, \varphi \, dx\, dy = 0$, by distribution derivatives definition. Since φ is any element of $\mathscr{D}$, this implies that $\Delta^2 u = 0$. The tempered distribution u is then polyharmonic and we conclude that u is a polynomial (cf. Schwartz [16]). Observing that the polynomials of $D^{-2}L^2$ are elements of $\mathscr{P}_1$ and using the equalities $u(X_i) = 0$, $i = 1, 2, 3$, we obtain $u = 0$. This completes the proof.

We deduce

PROPOSITION 2.2.

$$\int_{\mathbb{R}^2} \partial^2_{ab} u \, \partial^2_{cd} v \, dx\, dy = \int_{\mathbb{R}^2} \partial^2_{ad} u \, \partial^2_{cb} v \, dx\, dy,$$

$$\forall u, v \in D^{-2}L^2, \qquad \forall a, b, c, d \in \{x, y\}.$$

Proof. By a double integration by parts, the equality is true for u and $v \in \mathscr{D}$. Proposition 2.1 implies the result.

Let $\mathscr{X} = D^{-2}L^2(\mathbb{R}^2) \times D^{-2}L^2(\mathbb{R}^2)$, $V = (u, v) \in \mathscr{X}$ and α, β two positive constants.

We define the functional $J_{\alpha,\beta}: \mathscr{X} \to \mathbb{R}$,

$$J_{\alpha,\beta}(V) = \alpha \int_{\mathbb{R}^2} \|\nabla \operatorname{div} V\|^2 \, dx \, dy + \beta \int_{\mathbb{R}^2} \|\nabla \operatorname{rot} V\|^2 \, dx \, dy,$$

where

$$\|\nabla \operatorname{div} V\|^2 = (\partial_x \operatorname{div} V)^2 + (\partial_y \operatorname{div} V)^2,$$

$$\|\nabla \operatorname{rot} V\|^2 = (\partial_x \operatorname{rot} V)^2 + (\partial_y \operatorname{rot} V)^2.$$

We adopt the notations $D(V, V')$ for the semi-scalar product $\int_{\mathbb{R}^2} (\nabla \operatorname{div} V \cdot \nabla \operatorname{div} V') \, dx \, dy$ and $R(V, V')$ for $\int_{\mathbb{R}^2} (\nabla \operatorname{rot} V \cdot \nabla \operatorname{rot} V') \, dx \, dy$; the quadratic form $D(V, V)$ will be denoted $D(V)$ and $R(V, V)$, $R(V)$; $(\cdot)$ is the usual euclidean scalar product in $\mathbb{R}^2$ and $\| \ \|$ the associated norm.

PROPOSITION 2.3. *For each $V = (u, v)$ and $V' = (u', v') \in \mathscr{X}$,*

$$D(V, V') + R(V, V') = ((u \mid u')) + ((v \mid v')).$$

Proof. From Proposition 2.2, the terms $\int_{\mathbb{R}^2} \partial_{xx} u \, \partial_{xy} v' \, dx \, dy$, $\int_{\mathbb{R}^2} \partial_{xy} v \, \partial_{xx} u' \, dx \, dy$, $\int_{\mathbb{R}^2} \partial_{xy} u \, \partial_{yy} v' \, dx \, dy$, and $\int_{\mathbb{R}^2} \partial_{yy} v \, \partial_{xy} u' \, dx \, dy$ of $D(V, V')$ cancel each other out with corresponding terms of $R(V, V')$.

Proposition 2.3 implies that

$$J_{\alpha,\alpha}(V) = \alpha[D(V) + R(V)] = \alpha[((u \mid u)) + ((v \mid v))].$$

Let $X_i = (x_i, y_i)$, $i = 1, ..., N$, a set of N distinct points in $\mathbb{R}^2$ containing three non-aligned points, and $V_i = (u_i, v_i)$, N given couples in $\mathbb{R}^2$.

We define the minimization problem $P_{\alpha,\beta}$:

$$P_{\alpha,\beta} \begin{cases} \operatorname{Min} J_{\alpha,\beta}(V) \\ V \in \mathscr{X} \qquad \text{and} \qquad V(X_i) = V_i. \end{cases}$$

Remark 1. In fact, the solution of $P_{\alpha,\beta}$ only depends on the ratio $\rho = \alpha/\beta$.

Remark 2. If $\alpha = \beta$, Proposition 2.3 shows that $P_{\alpha,\alpha}$ splits into two separate problems, one for each of the components:

$$P_u \begin{cases} \operatorname{Min}((u \mid u)) \\ u \in D^{-2}L^2 \qquad \text{and} \qquad u(X_i) = u_i \end{cases}$$

and

$$P_v \begin{cases} \mathrm{Min}((v \mid v)) \\ v \in D^{-2}L^2 \qquad \text{and} \qquad v(X_i) = v_i, \end{cases}$$

which solutions are thin-plate splines. If $\alpha \neq \beta$, a coupling between u and v is introduced.

Remark 3. The choice of $J_{\alpha,\beta}$ follows naturally from Proposition 2.3. The weights α, β ($\alpha \neq \beta$) are intended to control the relative magnitude of $D(V)$ and $R(V)$.

Remark 4. The formal decomposition of the vector field V into rotational and divergent parts from a velocity potential χ and a stream function ψ (which is known, in fluid mechanics, as the Cauchy–Helmholtz Theorem) asserts

$$\begin{cases} u = \partial_x \chi - \partial_y \psi \\ v = \partial_y \chi + \partial_x \psi \end{cases}$$

(in vector notation, $V = \nabla\chi + \mathrm{Rot}\,\psi$, $\nabla\chi = (\partial_x \chi, \partial_y \chi)$, $\mathrm{Rot}\,\psi = (-\partial_y \psi, \partial_x \psi)$). Note that χ and ψ are solutions of the Poisson equations $\Delta\chi = \mathrm{div}\, V$ and $\Delta\psi = \mathrm{rot}\, V$. It is possible to define a minimization problem for (χ, ψ),

$$\begin{cases} \mathrm{Min}\, \alpha \int_{\mathbb{R}^2} \|\nabla\Delta\chi\|^2 \, dx\, dy + \beta \int_{\mathbb{R}^2} \|\nabla\Delta\psi\|^2 \, dx\, dy \\ \chi, \psi \in D^{-3}L^2 \qquad \text{and} \qquad \nabla\chi(X_i) + \mathrm{Rot}\,\psi(X_i) = V_i, \\ \chi(0) = 0, \qquad \psi(0) = 0, \end{cases}$$

from which u, v are deduced by differentiation.

The order of the Beppo–Levi space ($D^{-3}L^2$) is raised to 3 in order to define continuous functionals in the equalities constraints. The coupling between u and v is now introduced by the measurement constraints $\nabla\chi(X_i) + \mathrm{Rot}\,\psi(X_i) = V_i$. The indeterminacy of constants on χ and ψ is fixed by arbitrary values ($\chi(0) = 0$ and $\psi(0) = 0$).

However, the couples $(\chi, \psi) \in D^{-3}L^2 \times D^{-3}L^2$ that are solutions of this minimization problem are not unique: the condition $(\chi, \psi) \in \mathscr{P}_2 \times \mathscr{P}_2$, $\nabla\chi(X_i) + \mathrm{Rot}\,\psi(X_i) = 0$, $i = 1, ..., N$, and $\chi(0) = 0$, $\psi(0) = 0 \Rightarrow \chi = 0$ and $\psi = 0$ ($\mathscr{P}_2$ is the set of polynomials defined on $\mathbb{R}^2$ of degree 2) is not satisfied (cf. general spline theorem condition, cf. Laurent [12]). If $\Delta\chi$ is replaced by $\mathrm{div}\, V$ and $\Delta\psi$ by $\mathrm{rot}\, V$ in the functional above, and if the constraints are expressed on V ($V(X_i) = V_i$)), we get the problem $P_{\alpha,\beta}$.

3. Existence and Uniqueness of the Solution of $P_{\alpha,\beta}$

Consider the spaces $\mathscr{X}$ (defined above), $\mathscr{Y}=(L^2(\mathbb{R}^2))^4$, $\mathscr{Z}=\mathbb{R}^{2N}$, and the linear applications $T: \mathscr{X}\to\mathscr{Y}$ and $A: \mathscr{X}\to\mathscr{Z}$ defined by

$$T(V)=(\partial_x \operatorname{div} V, \partial_y \operatorname{div} V, \partial_x \operatorname{rot} V, \partial_y \operatorname{rot} V),$$

$$A(V)=(u(X_1), ..., u(X_N), v(X_1), ..., v(X_N)).$$

$\mathscr{X}$ is equipped with the scalar product

$$\langle V_1 \mid V_2\rangle=(u_1 \mid u_2)+(v_1 \mid v_2),$$

and $\mathscr{Y}$ with the scalar product

$$\alpha\int_{\mathbb{R}^2}(e_1 e_2+f_1 f_2)\,dx\,dy+\beta\int_{\mathbb{R}^2}(g_1 g_2+h_1 h_2)\,dx\,dy,$$

where $(e_1, f_1, g_1, h_1)\in\mathscr{Y}$, $(e_2, f_2, g_2, h_2)\in\mathscr{Y}$, and α, β are fixed positive constants. $\mathscr{Z}$ is equipped with the usual euclidean scalar product.

We can now state the main proposition.

PROPOSITION 3.1. *The linear applications T and A verify*:

(a) *T and A are continuous,*

(b) $\operatorname{Ker} T=\mathscr{P}_1\times\mathscr{P}_1$,

(c) $\operatorname{Ker} T\cap\operatorname{Ker} A=\{0\}$,

(d) *A is surjective,*

(e) $\operatorname{Im} T$ *is closed.*

Proof. (a) From the definitions of the norms in the spaces $\mathscr{X}$ and $\mathscr{Y}$ and Proposition 2.3, T is clearly continuous. Continuity of A follows from the continuous embedding $D^{-2}L^2 \hookrightarrow \mathscr{C}^0$ (Sobolev-type lemma; cf. Necas [14]).

(b) Proposition 2.3 shows that

$$J_{\alpha,\beta}(V)=0 \Rightarrow \partial^2_{xx}u=0, \qquad \partial^2_{xy}u=0, \qquad \partial^2_{yy}u=0$$

and $\partial^2_{xx}v=0$, $\partial^2_{xy}v=0$, $\partial^2_{yy}v=0$, which gives the result.

(c) Results from the hypothesis on the points X_i and (b).

(d) Let the classical function $f_R(\|X\|)\in\mathscr{D}$ be defined by

$$f_R(\|X\|)=\begin{cases}\exp(1)\exp\left(-\dfrac{R^2}{R^2-\|X\|^2}\right) & \text{if } \|X\|<R \\ 0 & \text{otherwise}\end{cases} \qquad (R>0).$$

Since the points X_i, $i=1, ..., N$, are distinct, we can take R_i, $i=1, ..., N$, such that the open balls $B(X_i, R_i)$ are disjoint. The functions $f_i(X) = f_{R_i}(\|X - X_i\|)$ verify $f_i(X_j) = \delta_{ij}$, $i, j = 1, ..., N$. We define

$$u(X) = \sum_{i=1}^{N} u_i f_i(x) \qquad \text{and} \qquad v(X) = \sum_{i=1}^{N} v_i f_i(X).$$

The function $V(X) = (u(X), v(X))$ is an element of $\mathscr{X}$ and satisfies the interpolation conditions $V(X_i) = V_i$.

(e) Let $(e, f, g, h) \in \mathscr{Y}$ and (V_n) be a sequence in $\mathscr{X}$ such that $\operatorname{Lim} T(V_n) = (e, f, g, h)$. $T(V_n)$ is a Cauchy sequence in $\mathscr{Y}$. Proposition 2.3 shows that $(\partial^2_{xx} u_n)$, $(\partial^2_{xy} u_n)$, $(\partial^2_{yy} u_n)$, $(\partial^2_{xx} v_n)$, $(\partial^2_{xy} v_n)$, $(\partial^2_{yy} v_n)$ are also Cauchy sequences in L^2 and therefore converge.

Let l, m, n and $l', m', n' \in L^2$ such that

$$\begin{cases} \operatorname{Lim}(\partial^2_{xx} u_n) = l, & \operatorname{Lim}(\partial^2_{xy} u_n) = m, & \operatorname{Lim}(\partial^2_{yy} u_n) = n \\ \operatorname{Lim}(\partial^2_{xx} v_n) = l', & \operatorname{Lim}(\partial^2_{xy} v_n) = m', & \operatorname{Lim}(\partial^2_{yy} v_n) = n'. \end{cases} \tag{3.1}$$

Since the compatibility conditions $(\partial^2_{xy} l = \partial^2_{xx} m, ...)$ are satisfied, we may conclude (cf. Schwartz [16, p. 59], and Benbourhim [2]) that there exist u and $v \in D^{-2}L^2$ such that

$$\begin{cases} \partial^2_{xx} u = l, & \partial^2_{xy} u = m, & \partial^2_{yy} u = n \\ \partial^2_{xx} v = l', & \partial^2_{xy} v = m', & \partial^2_{yy} v = n'. \end{cases} \tag{3.2}$$

But $\operatorname{Lim}(\partial^2_{xx} u_n + \partial^2_{xy} v_n) = e \Rightarrow \partial^2_{xx} u + \partial^2_{xy} v = e$, $\operatorname{Lim}(\partial^2_{xy} u_n + \partial^2_{yy} v_n) = f \Rightarrow \partial^2_{xy} u + \partial^2_{yy} v = f$, $\operatorname{Lim}(\partial^2_{xx} v_n - \partial^2_{xy} u_n) = g \Rightarrow \partial^2_{xx} v - \partial^2_{xy} u = g$, and $\operatorname{Lim}(\partial^2_{xy} v_n - \partial^2_{yy} u_n) = h \Rightarrow \partial^2_{xy} v - \partial^2_{yy} u = h$. This shows that $(e, f, g, h) \in \operatorname{Im} T$.

Proposition 3.1 implies the theorem:

THEOREM 3.2. (a) *The problem* $P_{\alpha,\beta}$ *(with fixed* $\alpha, \beta > 0$*) admits a unique solution* $V_{\alpha,\beta}$ *in* $\mathscr{X}$.

(b) $V_{\alpha,\beta} \in \mathscr{X}$ *such that* $V_{\alpha,\beta}(X_i) = V_i$ *is the solution of* $P_{\alpha,\beta}$ *if and only if there exist* a_i, $b_i \in \mathbb{R}$, $i = 1, ..., N$, *such that*

$$\alpha D(V_{\alpha,\beta}, V') + \beta R(V_{\alpha,\beta}, V') = \sum_{i=1}^{N} a_i u'(X_i) + \sum_{i=1}^{N} b_i v'(X_i), \qquad \forall V' = (u', v') \in \mathscr{X}. \tag{3.3}$$

The coefficients a_i, b_i, $i = 1, ..., N$, *are unique.*

Proof. (a) and (b) result from Proposition 3.1 and general spline theorems (cf. Laurent [12]).

4. Expression of the Solution $V_{\alpha,\beta}$

PROPOSITION 4.1. *For $V \in \mathscr{X}$, the following properties are equivalent*:

(1) *V verifies the equality*

$$\alpha D(V, V') + \beta R(V, V') = \sum_{i=1}^{N} a_i u'(X_i) + \sum_{i=1}^{N} b_i v'(X_i) \qquad \forall V' \in \mathscr{X}. \tag{4.1}$$

(2) *V verifies the system*

$$\begin{cases} \Delta(\alpha \partial_x \operatorname{div} V - \beta \partial_y \operatorname{rot} V) = \sum_{i=1}^{N} a_i \delta_{X_i} \\ \Delta(\alpha \partial_y \operatorname{div} V + \beta \partial_x \operatorname{rot} V) = \sum_{i=1}^{N} b_i \delta_{X_i} \end{cases} \tag{4.2}$$

in the distribution space (δ_{X_i} is the Dirac distribution at point X_i) and the coefficients a_i, b_i, $i = 1, ..., N$, satisfy the orthogonality conditions

$$\sum_{i=1}^{N} a_i p(X_i) = \sum_{i=1}^{N} b_i p(X_i) = 0, \qquad \forall p \in \mathscr{P}_1. \tag{4.3}$$

Proof. By taking $V' = (\varphi, 0)$ and $V' = (0, \varphi)$, $\forall \varphi \in \mathscr{D}$ in Eq. (4.1), and using the definition of distribution derivatives, we deduce the system (4.2). With $V' = (p, 0)$ and $V' = (0, p)$, $\forall p \in \mathscr{P}_1$, we deduce (4.3). Conversely, Eqs. (4.2) and (4.3) $\Rightarrow$ Eq. (4.1) for each $V' \in \mathscr{D} \times \mathscr{D}$ and $V' \in \mathscr{P}_1 \times \mathscr{P}_1$. Using Proposition 2.1 (density of $\mathscr{D} + \mathscr{P}_1$) and the continuous dependence on V' in the equality (4.1), we deduce the result.

Note that, for $\alpha = \beta$, we get the system

$$\begin{cases} \Delta^2 u_{\alpha,\alpha} = \dfrac{1}{\alpha} \sum_{i=1}^{N} a_i \delta_{X_i} \\ \Delta^2 v_{\alpha,\alpha} = \dfrac{1}{\alpha} \sum_{i=1}^{N} b_i \delta_{X_i}, \end{cases}$$

issued from a couple of Duchon's spline problems.

PROPOSITION 4.2. *The solution $V_H \in \mathscr{X}$ of the homogeneous system associated to Eqs.* (4.2) *are the polynomials $\mathscr{P}_1 \times \mathscr{P}_1$.*

Proof. A solution V_H of the homogeneous system associated to (4.2) verifies $\alpha D(V_H, V') + \beta R(V_H, V') = 0$, $\forall V' \in \mathscr{D} \times \mathscr{D}$. Since this equality is also verified with $V' \in \mathscr{P}_1 \times \mathscr{P}_1$, we deduce, by density (Proposition 2.1), that it is true for each $V' \in \mathscr{X}$. By taking $V' = V_H$, we deduce the result.

Consider $K(X)=\theta\,\|X\|^4\log\|X\|$ with $X\in\mathbb{R}^2$ and θ a real constant ($\theta=-1/2^7\pi$); K is a fundamental solution of the operator Δ^3, i.e., $\Delta^3 K=\delta$ (δ is the Dirac distribution at the origin).

Let μ be a compact support measure orthogonal to $\mathscr{P}_1$, i.e.,

$$\int_{\mathbb{R}^2} p\,d\mu=0, \qquad \forall p\in\mathscr{P}_1.$$

LEMMA 4.3. *For any compact support measure μ orthogonal to $\mathscr{P}_1$, $D^\gamma(\mu * K)\in D^{-2}L^2$, for each $|\gamma|=2$.*

Proof. We want to show that $D^\gamma(\mu * K)\in L^2$, for each $|\gamma|=4$, or equivalently, show that the Fourier transform of this function is an element of L^2. Following Schwartz (cf. [16]), we get $D^\gamma\widehat{(\mu * K)}=\xi^\gamma\hat{\mu}(\xi)(\tau_1\,\mathrm{Fp}(1/\|\xi\|^6)+\tau_2\Delta^2\delta)$, where $\xi=(\xi_1,\xi_2)$ is the Fourier space variable, τ_1 and τ_2 are two real constants, and Fp is the finite part of the distribution. It is easy to see that $\xi^\gamma\hat{\mu}(\xi)\Delta^2\delta=0$ for $|\gamma|\geqslant 3$ (cf. Duchon [8]). Now, in the unit ball $B(0,1)$, the inequality $|\hat{\mu}(\xi)|\leqslant\kappa_1\,\|\xi\|^2$ (κ_1 is a positive constant) is verified because μ is orthogonal to $\mathscr{P}_1$; we get the estimate $|\xi^\gamma\hat{\mu}(\xi)|/\|\xi\|^6\leqslant\kappa_1/|\xi\|^{4-|\gamma|}$ and hence, the integral converges for $|\gamma|>3$. In the complement of $B(0,1)$, the equality $|\hat{\mu}(\xi)|\leqslant\kappa_2$ (κ_2 is a positive constant) is verified because the Fourier transform of a bounded measure is a bounded function; we get the estimate $|\xi^\gamma\hat{\mu}(\xi)|/\|\xi\|^6\leqslant\kappa_2/\|\xi\|^{6-|\gamma|}$, and the integral converges for $|\gamma|<5$. This completes the proof.

It is now possible to state

THEOREM 4.4. *The solution $V_{\alpha,\beta}=(u_{\alpha,\beta},v_{\alpha,\beta})$ of the problem $P_{\alpha,\beta}$ admits a unique expression of the form*

$$\begin{aligned}
u_{\alpha,\beta}(X)=&\sum_{i=1}^{N}a_i\left(\frac{1}{\alpha}\,\partial^2_{xx}K(X-X_i)+\frac{1}{\beta}\partial^2_{yy}K(X-X_i)\right)\\
&+\sum_{i=1}^{N}b_i\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\partial^2_{xy}K(X-X_i)+p(X)\\
v_{\alpha,\beta}(X)=&\sum_{i=1}^{N}a_i\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\partial^2_{xy}K(X-X_i)\\
&+\sum_{i=1}^{N}b_i\left(\frac{1}{\alpha}\,\partial^2_{yy}K(X-X_i)+\frac{1}{\beta}\,\partial^2_{xx}K(X-X_i)\right)+q(X);
\end{aligned}$$

$a_i, b_i\in\mathbb{R}$, $i=1,\dots,N$, are given by the equality (3.3), *and $p(X)$, $q(X)\in\mathscr{P}_1$, $p(X)=c_1+c_2x+c_3y$, $q(X)=d_1+d_2x+d_3y$.*

The coefficients are obtained by solving the $(2N+6)\times(2N+6)$ linear system

$$\left(\frac{1}{\alpha}K_{\mathrm{D}}+\frac{1}{\beta}K_{\mathrm{R}}\right)\begin{pmatrix}a\\b\end{pmatrix}+P\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}u\\v\end{pmatrix}$$

$$P^{t}\begin{pmatrix}a\\b\end{pmatrix}=0,$$

with the $2N\times 2N$ matrices

$$K_{\mathrm{D}}=\begin{pmatrix}(\partial^2_{xx}K(X_i-X_j)) & (\partial^2_{xy}K(X_i-X_j))\\ (\partial^2_{xy}K(X_i-X_j)) & (\partial^2_{yy}K(X_i-X_j))\end{pmatrix}$$

and

$$K_{\mathrm{R}}=\begin{pmatrix}(\partial^2_{yy}K(X_i-X_j)) & (-\partial^2_{xy}K(X_i-X_j))\\ (-\partial^2_{xy}K(X_i-X_j)) & (\partial^2_{xx}K(X_i-X_j))\end{pmatrix},$$

and the $2N\times 6$ matrix

$$P=\begin{pmatrix}\begin{pmatrix}1 & x_1 & y_1\\ \vdots & \vdots & \vdots\\ 1 & x_N & y_N\end{pmatrix} & 0\\ 0 & \begin{pmatrix}1 & x_1 & y_1\\ \vdots & \vdots & \vdots\\ 1 & x_N & y_N\end{pmatrix}\end{pmatrix};$$

$$\begin{pmatrix}a\\b\end{pmatrix}\ \textit{stands for}\ \begin{pmatrix}a_1\\ \vdots\\ a_N\\ b_1\\ \vdots\\ b_N\end{pmatrix},\qquad \begin{pmatrix}c\\d\end{pmatrix}\ \textit{for}\ \begin{pmatrix}c_1\\ c_2\\ c_3\\ d_1\\ d_2\\ d_3\end{pmatrix},\qquad \textit{and}$$

$$\begin{pmatrix}u\\v\end{pmatrix}\ \textit{for}\ \begin{pmatrix}u_1\\ \vdots\\ u_N\\ v_1\\ \vdots\\ v_N\end{pmatrix}.$$

P^t is the transpose of P.

Proof. $V_{\alpha,\beta}$ verifies the system (4.2) with a_i, b_i, $i=1, ..., N$, given by Theorem 3.2. By taking the Fourier transform $\widehat{V_{\alpha,\beta}}$ of $V_{\alpha,\beta}$ ($\mathscr{X}$ is contained in the space of tempered distributions), we get the system

$$(\xi_1^2+\xi_2^2)\begin{pmatrix} (\alpha\xi_1^2+\beta\xi_2^2) & ((\alpha-\beta)\xi_1\xi_2) \\ ((\alpha-\beta)\xi_1\xi_2) & (\alpha\xi_2^2+\beta\xi_1^2) \end{pmatrix}\begin{pmatrix} \widehat{u_{\alpha,\beta}}(\xi_1, \xi_2) \\ \widehat{v_{\alpha,\beta}}(\xi_1, \xi_2) \end{pmatrix}$$
$$=\frac{1}{(2j\pi)^4}\sum_{i=1}^{N} \exp(-2j\pi(X_i\cdot\xi))\begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

The solution of the system is

$$\widehat{u_{\alpha,\beta}}(\xi)=\frac{1}{(2j\pi)^4}\sum_{i=1}^{N} \exp(-2j\pi(X_i\cdot\xi))$$
$$\left(a_i\left(\frac{1}{\alpha}Fp\frac{\xi_1^2}{(\xi_1^2+\xi_2^2)^3}+\frac{1}{\beta}Fp\frac{\xi_2^2}{(\xi_1^2+\xi_2^2)^3}\right)\right.$$
$$\left.+b_i\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)Fp\frac{\xi_1\xi_2}{(\xi_1^2+\xi_2^2)^3}\right)$$
$$v_{\alpha,\beta}(\xi)=\frac{1}{(2j\pi)^4}\sum_{i=1}^{N} \exp(-2j\pi(X_i\cdot\xi))\left(a_i\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)Fp\frac{\xi_1\xi_2}{(\xi_1^2+\xi_2^2)^3}\right.$$
$$\left.+b_i\left(\frac{1}{\alpha}Fp\frac{\xi_2^2}{(\xi_1^2+\xi_2^2)^3}+\frac{1}{\beta}Fp\frac{\xi_1^2}{(\xi_1^2+\xi_2^2)^3}\right)\right).$$

The inverse Fourier transform of $(\widehat{u_{\alpha,\beta}}, \widehat{v_{\alpha,\beta}})$, after the polynomial part is removed (cf. Schwartz [16]), gives $V_K=(u_K, v_K)$ of the form

$$u_K(X)=\sum_{i=1}^{N} a_i\left(\frac{1}{\alpha}\partial_{xx}^2 K(X-X_i)+\frac{1}{\beta}\partial_{yy}^2 K(X-X_i)\right)$$
$$+\sum_{i=1}^{N} b_i\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\partial_{xy}^2 K(X-X_i)$$
$$v_K(X)=\sum_{i=1}^{N} a_i\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\partial_{xy}^2 K(X-X_i)$$
$$+\sum_{i=1}^{N} b_i\left(\frac{1}{\alpha}\partial_{yy}^2 K(X-X_i)+\frac{1}{\beta}\partial_{xx}^2 K(X-X_i)\right).$$

It is easy to check that this function satisfies Eqs. (4.2). Since Eq. (4.3) is verified by the coefficients a_i, b_i, $i=1, ..., N$, Lemma 4.3, implies that $V_K\in\mathscr{X}$.

Suppose now that $V_{\alpha,\beta}-V_K$ verifies the homogeneous system associated with Eqs. (4.2). From Proposition 4.2, we see that $V_{\alpha,\beta}-V_K\in\mathscr{P}_1\times\mathscr{P}_1$ and we deduce for $V_{\alpha,\beta}$ the expression of Theorem 4.4. The uniqueness of the expression of $V_{\alpha,\beta}$ follows from Theorem 3.2.

The linear system verified by (a, b, c, d) is obtained by taking the successive equalities

$$u_{\alpha,\beta}(X_i) = u_i$$
$$v_{\alpha,\beta}(X_i) = v_i$$

from interpolation constraints ($2N$ first lines of the system) and

$$\sum_{i=1}^{N} a_i = 0, \qquad \sum_{i=1}^{N} a_i x_i = 0, \qquad \sum_{i=1}^{N} a_i y_i = 0,$$
$$\sum_{i=1}^{N} b_i = 0, \qquad \sum_{i=1}^{N} b_i x_i = 0, \qquad \sum_{i=1}^{N} b_i y_i = 0,$$

obtained from Eq. (4.3) (six last lines of the system).

The existence and uniqueness of the expression of $V_{\alpha,\beta}$ shows that the system is invertible. This completes the proof.

Using the notations of Theorem 4.4, we state

PROPOSITION 4.5. *The solution $V_{\alpha,\beta}$ of the problem $P_{\alpha,\beta}$ verifies the equalities*

$$\text{(a)} \quad J_{\alpha,\beta}(V_{\alpha,\beta}) = \frac{1}{\alpha}\begin{pmatrix} a \\ b \end{pmatrix}^t K_{\mathrm{D}} \begin{pmatrix} a \\ b \end{pmatrix} + \frac{1}{\beta}\begin{pmatrix} a \\ b \end{pmatrix}^t K_{\mathrm{R}} \begin{pmatrix} a \\ b \end{pmatrix},$$

$$\text{(b)} \quad D(V_{\alpha,\beta}) = \frac{1}{\alpha^2}\begin{pmatrix} a \\ b \end{pmatrix}^t K_{\mathrm{D}} \begin{pmatrix} a \\ b \end{pmatrix},$$

$$\text{(c)} \quad R(V_{\alpha,\beta}) = \frac{1}{\beta^2}\begin{pmatrix} a \\ b \end{pmatrix}^t K_{\mathrm{R}} \begin{pmatrix} a \\ b \end{pmatrix},$$

where $\binom{a}{b}$ is the solution of the linear system given by Theorem 4.4 (*or equivalently, given by Eq.* (3.3)).

Proof. (a) By taking $V' = V_{\alpha,\beta}$ in Eq. (3.3) and using the expression of $V_{\alpha,\beta}$ given by Theorem 4.4, the result follows.

(b) Let $V_{\mathrm{D}} = (u_{\mathrm{D}}, v_{\mathrm{D}})$ defined by

$$u_{\mathrm{D}}(X) = \frac{1}{\alpha}\left(\sum_{i=1}^{N} a_i \partial^2_{xx} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{xy} K(X - X_i)\right)$$
$$v_{\mathrm{D}}(X) = \frac{1}{\alpha}\left(\sum_{i=1}^{N} a_i \partial^2_{xy} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{yy} K(X - X_i)\right).$$

By taking $V' = V_D$ in Eq. (3.3) and observing that $D(V_{\alpha,\beta}, V_D) = D(V_{\alpha,\beta}, V_{\alpha,\beta}) = D(V_{\alpha,\beta})$ and that $R(V_{\alpha,\beta}, V_D) = 0$, we get the result.

(c) The proof is the same, using $V_R = (u_R, v_R)$ defined by

$$u_R(X) = \frac{1}{\beta}\left(\sum_{i=1}^{N} a_i \partial_{yy}^2 K(X - X_i) - \sum_{i=1}^{N} b_i \partial_{xy}^2 K(X - X_i)\right)$$

$$v_R(X) = \frac{1}{\beta}\left(-\sum_{i=1}^{N} a_i \partial_{xy}^2 K(X - X_i) + \sum_{i=1}^{N} b_i \partial_{xx}^2 K(X - X_i)\right).$$

From equalities (b) and (c), it becomes possible to calculate effectively the quantities $D(V_{\alpha,\beta})$ and $R(V_{\alpha,\beta})$; thus, by modifying the ratio $\rho = \alpha/\beta$ and using the monotonicity properties of $D(V_\rho)$ and $R(V_\rho)$ (cf. remarks of Proposition 5.3.1, Section 5.3), we get a control on the ratio $D(V_\rho)/R(V_\rho)$.

PROPOSITION 4.6. *The matrices* $((1/\alpha)K_D + (1/\beta)K_R)$, K_D, *and* K_R *are positive definite on the subspace* $\{\binom{a}{b} / P^t\binom{a}{b} = 0\}$.

Proof. Let any $\binom{a}{b}$ such that $P^t\binom{a}{b} = 0$ (a_i, b_i, $i = 1, ..., N$, verify Eq. 4.3). Consider $V_{D+R} \in \mathscr{X}$ of the form

$$u_{D+R}(X) = \sum_{i=1}^{N} a_i \left(\frac{1}{\alpha}\partial_{xx}^2 K(X - X_i) + \frac{1}{\beta}\partial_{yy}^2 K(X - X_i)\right) + \sum_{i=1}^{N} b_i \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\partial_{xy}^2 K(X - X_i)$$

$$v_{D+R}(X) = \sum_{i=1}^{N} a_i \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\partial_{xy}^2 K(X - X_i) + \sum_{i=1}^{N} b_i \left(\frac{1}{\alpha}\partial_{yy}^2 K(X - X_i) + \frac{1}{\beta}\partial_{xx}^2 K(X - X_i)\right).$$

V_{D+R} satisfies Eqs. (4.2) and consequently by taking $V' = V_{D+R}$ in Eq. (4.1), we get

$$\alpha D(V_{D+R}) + \beta R(V_{D+R}) = \frac{1}{\alpha}\begin{pmatrix}a\\b\end{pmatrix}^t K_D \begin{pmatrix}a\\b\end{pmatrix} + \frac{1}{\beta}\begin{pmatrix}a\\b\end{pmatrix}^t K_R \begin{pmatrix}a\\b\end{pmatrix}.$$

This shows that the matrix $(1/\alpha)K_D + (1/\beta)K_R$ is positive. Suppose now that

$$\frac{1}{\alpha}\begin{pmatrix}a\\b\end{pmatrix}^t K_D \begin{pmatrix}a\\b\end{pmatrix} + \frac{1}{\beta}\begin{pmatrix}a\\b\end{pmatrix}^t K_R \begin{pmatrix}a\\b\end{pmatrix} = 0.$$

From the above equality, we deduce that $V_{D+R} = (p, q)$, p, $q \in \mathscr{P}_1$. Again, Eq. (4.2) implies that $\sum_{i=1}^{N} a_i \delta_{X_i} = 0$ and $\sum_{i=1}^{N} b_i \delta_{X_i} = 0$. By using $f_i(X) \in \mathscr{D}$

defined as in Proposition 3.1(d), we deduce that $a_i = b_i = 0$, $i = 1, ..., N$. The matrix $(1/\alpha)K_D + (1/\beta)K_R$ is therefore positive definite in the subspace $\{\binom{a}{b}/P^t\binom{a}{b} = 0\}$.

To show the same for the matrix K_D, we consider $V_D \in \mathscr{X}$ defined as in Proposition 4.5(b) ($V_{D+R} = V_D + V_R$). By using again Eq. (4.1) (true for V_{D+R}) with $V' = V_D$ and observing that $D(V_{D+R}, V_D) = D(V_D, V_D) = D(V_D)$ and $R(V_{D+R}, V_D) = 0$, we get $D(V_D) = (1/\alpha^2)\binom{a}{b}^t K_D \binom{a}{b}$ and therefore K_D is positive. Suppose now that $\binom{a}{b}^t K_D \binom{a}{b} = 0$. This implies that $D(V_D) = 0$ and from $R(V_D) = 0$, we conclude that $V_D = (p, q)$, $p, q \in \mathscr{P}_1$. With the use of the property of $K(X)$, we obtain the equality $\Delta^2 \operatorname{div}(V_D) = \sum_{i=1}^N a_i \delta'_{x,X_i} + \sum_{i=1}^N b_i \delta'_{y,X_i} = 0$, where δ'_{x,X_i} is the derivative of the Dirac distribution along x at point X_i and δ'_{y,X_i} the derivative along y.

To show that $a_i = b_i = 0$, $i = 1, ..., N$, we construct a family of functions $\widetilde{f_i}(X) \in \mathscr{D}$ such that $\nabla\widetilde{f_i}(X_i) = (a_i, b_i)$ and $\nabla\widetilde{f_i}(X_j) = 0$, $j \neq i$: let $f_R(\|X\|)$ be defined as in Proposition 3.1(d) and $\widetilde{X_i} = X_i + t_i(a_i, b_i)$, $t_i \in \mathbb{R}$. R_i and $\widetilde{X_i}$, $i = 1, ..., N$, are taken such that the open balls $B(\widetilde{X_i}, R_i)$ are disjoint and $X_i \in B(\widetilde{X_i}, R_i)$, with $X_i \neq \widetilde{X_i}$. It is easy to verify that $f_{R_i}(\|X - \widetilde{X_i}\|)$ satisfy the equality $\nabla f_{R_i}(\|X_i - \widetilde{X_i}\|) = k_i(a_i, b_i)$ (k_i constant $\neq 0$); the functions $\widetilde{f_i}(X) = (1/k_i) f_{R_i}(\|X - \widetilde{X_i}\|)$ satisfy the above property.

For the matrix K_R, the proof is similar.

5. Limit Problems ($\rho \to 0$ and $\rho \to +\infty$)

Following the Remark 1 in Section 2, we consider $J_\rho(V) = \rho D(V) + R(V)$ instead of $J_{\alpha,\beta}$. We want to show that the solution V_ρ of the problem P_ρ admits two limits as $\rho \to 0$ and $\rho \to +\infty$.

Consider the problems

$$P_0 \begin{cases} \text{Min } D(V) \\ V \in \mathscr{X}, \quad R(V) = 0 \quad \text{and} \quad V(X_i) = V_i \end{cases}$$

and

$$P_{+\infty} \begin{cases} \text{Min } R(V) \\ V \in \mathscr{X}, \quad D(V) = 0 \quad \text{and} \quad V(X_i) = V_i. \end{cases}$$

We show that P_0 and $P_{+\infty}$ admit unique solutions denoted by V_0 and $V_{+\infty}$, respectively, and $\text{Lim}_{\rho \to 0} V_\rho = V_0$, $\text{Lim}_{\rho \to +\infty} V_\rho = V_{+\infty}$.

5.1. *Existence and Uniqueness of the Solution of P_0 (and $P_{+\infty}$)*

We define the subspace $\widetilde{\mathscr{X}_D} = \{V \in \mathscr{X} / R(V) = 0\}$ of $\mathscr{X}$. From Proposition 2.3, we see that the semi-scalar product $D(V, V')$ is continuous for the norm $\| \ \|_{\mathscr{X}}$ ($\| \ \|_{\mathscr{X}}$ is the norm of $\mathscr{X}$ associated with $\langle \mid \rangle$); the quadratic form $D(V)$ is then continuous and $\widetilde{\mathscr{X}_D}$ is a closed subset of $\mathscr{X}$. Therefore, $\widetilde{\mathscr{X}_D}$ is a Hilbert space for the scalar product $\langle \mid \rangle$.

We define the spaces $\mathscr{Y} = (L^2(\mathbb{R}^2))^2$, $\mathscr{Z} = \mathbb{R}^{2N}$, and the linear applications $T: \widetilde{\mathscr{X}_D} \to \mathscr{Y}$ and $A: \widetilde{\mathscr{X}_D} \to \mathscr{Z}$ by $T(V) = (\partial_x \operatorname{div} V, \ \partial_y \operatorname{div} V)$ and $A(V) = (u(X_1), ..., u(X_N), v(X_1), ..., v(X_N))$. $\mathscr{Y}$ is equipped with the scalar product $\int_{\mathbb{R}^2} (e_1 e_2 + f_1 f_2) \, dx \, dy$, $(e_1, f_1) \in \mathscr{Y}$, $(e_2, f_2) \in \mathscr{Y}$; $\mathscr{Z}$ is equipped with the usual euclidean scalar product.

We state the theorem relative to problem P_0.

THEOREM 5.1.1. (a) *The problem P_0 admits a unique solution V_0 in $\widetilde{\mathscr{X}_D}$.*

(b) *$V_0 \in \widetilde{\mathscr{X}_D}$ such that $V_0(X_i) = V_i$ is the solution of P_0 if and only if there exist $a_i, b_i \in \mathbb{R}$, $i = 1, ..., N$, such that*

$$D(V_0, V') = \sum_{i=1}^{N} a_i u'(X_i) + \sum_{i=1}^{N} b_i v'(X_i), \tag{5.1}$$

$\forall V' = (u', v') \in \widetilde{\mathscr{X}_D}$. The coefficients a_i, b_i, $i = 1, ..., N$, are unique.

Proof. The points (a), (b), and (c) of Proposition 3.1 are easily verified. The main differences come from point (d) and (e).

(d) A is surjective: As for Proposition 4.6, we construct a family of functions $\widetilde{f_i(X)} \in \mathscr{D}$ such that $\nabla \widetilde{f_i}(X_i) = (u_i, v_i)$ and $\nabla \widetilde{f_i}(X_j) = 0$, $j \neq i$. The function $V(X) = \sum_{i=1}^{N} \nabla \widetilde{f_i}(X)$ is an element of $\mathscr{X}$ and satisfies $R(V) = 0$ and $V(X_i) = V_i$, $i = 1, ..., N$.

(e) $\operatorname{Im} T$ is closed: Consider a sequence (V_n) in $\widetilde{\mathscr{X}_D}$ such that $\operatorname{Lim} T(V_n) = (e, f)$, $(e, f) \in \mathscr{Y}$. From Proposition 2.3, we observe that $[((u \mid u)) + ((v \mid v))] = \|T(V)\|_{\mathscr{Y}}^2$ for each $V \in \widetilde{\mathscr{X}_D}$. Thus $\partial_{xx}^2 u_n$, $\partial_{xy}^2 u_n$, $\partial_{yy}^2 u_n$ and $\partial_{xx}^2 v_n$, $\partial_{xy}^2 v_n$, $\partial_{yy}^2 v_n$ are Cauchy sequences in L^2 and therefore converge. We deduce the existence of l, m, n, $l', m' n' \in L^2$ and $u, v \in \mathscr{X}$ verifying Eqs. (3.1) and (3.2). But $R(V_n) = 0$, $\forall n$, implies that $\partial_{xx}^2 v_n - \partial_{xy}^2 u_n \to 0$ and $\partial_{xy}^2 v_n - \partial_{yy}^2 u_n \to 0$. This shows that $(u, v) \in \widetilde{\mathscr{X}_D}$ and $(e, f) \in \operatorname{Im} T$.

From these properties and general spline theorems, we deduce the result.

For the problem $P_{+\infty}$, we get a symmetric result; let us define $\widetilde{\mathscr{X}_R} = \{V \in \mathscr{X} / D(V) = 0\}$.

THEOREM 5.1.2. (a) *The problem* $P_{+\infty}$ *admits a unique solution* $V_{+\infty}$ *in* $\mathscr{X}_{\mathrm{R}}$.

(b) $V_{+\infty} \in \widetilde{\mathscr{X}_{\mathrm{R}}}$ *such that* $V_{+\infty}(X_i) = V_i$ *is the solution of* $P_{+\infty}$ *if and only if there exist* $a_i, b_i \in \mathbb{R}$, $i = 1, ..., N$, *such that*

$$R(V_{+\infty}, V') = \sum_{i=1}^{N} a_i u'(X_i) + \sum_{i=1}^{N} b_i v'(X_i) \tag{5.2}$$

$\forall V' = (u', v') \in \widetilde{\mathscr{H}_{\mathrm{R}}}$. *The coefficients* a_i, b_i, $i = 1, ..., N$, *are unique.*

5.2. *Expression of the Solutions* V_0 *and* $V_{+\infty}$

THEOREM 5.2.1. *The solution* $V_0 = (u_0, v_0)$ *of the problem* P_0 *admits a unique expression of the form*

$$u_0(X) = \sum_{i=1}^{N} a_i \partial^2_{xx} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{xy} K(X - X_i) + p(X)$$

$$v_0(X) = \sum_{i=1}^{N} a_i \partial^2_{xy} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{yy} K(X - X_i) + q(X),$$

with $a_i, b_i \in \mathbb{R}$, $i = 1, ..., N$, *given by* Eq. (5.1) *and* $p(X)$, $q(X) \in \mathscr{P}_1$, $p(X) = c_1 + c_2 x + c_3 y$, $q(X) = d_1 + d_2 x + d_3 y$.

The coefficients are obtained by solving the $(2N+6) \times (2N+6)$ *linear system*

$$K_{\mathrm{D}} \begin{pmatrix} a \\ b \end{pmatrix} + P \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$P^t \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

Proof. By taking $V' = \nabla\varphi$, $\varphi \in \mathscr{D}$ in the equality (5.1) and observing that $R(V') = 0$, we get

$$\Delta^2 \operatorname{div} V_0 = \sum_{i=1}^{N} a_i \delta'_{x, X_i} + \sum_{i=1}^{N} b_i \delta'_{y, X_i}. \tag{5.3}$$

From the property of $K(X)$, it is easy to verify that $V_{\mathrm{D}} = (u_{\mathrm{D}}, v_{\mathrm{D}})$, defined by

$$u_{\mathrm{D}}(X) = \sum_{i=1}^{N} a_i \partial^2_{xx} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{xy} K(X - X_i)$$

$$v_{\mathrm{D}}(X) = \sum_{i=1}^{N} a_i \partial^2_{xy} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{yy} K(X - X_i),$$

satisfies (5.3). Thus $V_0 - V_D$ satisfies the homogeneous system $\Delta^2 \operatorname{div}(V_0 - V_D) = 0$ associated to (5.3). Using the same arguments as for Proposition 2.1, we deduce that $\operatorname{div}(V_0 - V_D)$ is a constant. This implies that $D(V_0 - V_D) = 0$, and since $R(V_0 - V_D) = 0$, we get $V_0 = V_D + (p, q)$, $p, q \in \mathscr{P}_1$. The uniqueness of this expression is guaranteed by Theorem 5.1.1(b). The linear system is deduced in the same way as for Theorem 4.4.

For $P_{+\infty}$, the theorem gives a solution $V_{+\infty} = (u_{+\infty}, v_{+\infty})$ of the form

$$u_{+\infty}(X) = \sum_{i=1}^{N} a_i \partial^2_{yy} K(X - X_i) - \sum_{i=1}^{N} b_i \partial^2_{xy} K(X - X_i) + p(X)$$

$$v_{+\infty}(X) = -\sum_{i=1}^{N} a_i \partial^2_{xy} K(X - X_i) + \sum_{i=1}^{N} b_i \partial^2_{xx} K(X - X_i) + q(X),$$

with the associated linear system

$$K_R \begin{pmatrix} a \\ b \end{pmatrix} + P \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$P^t \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

5.3. *Limit of* V_ρ

In the scalar product $\langle \mid \rangle$ of $\mathscr{X}$, we consider that the three non-aligned points X_1, X_2, X_3 (cf. Section 2) are elements of the set of interpolation points; it follows that $V_\rho(X_i) = V_i$, $i = 1, 2, 3$, $\forall \rho \in [0, +\infty]$.

We state some immediate properties of the functions $\rho \to D(V_\rho)$ and $\rho \to R(V_\rho)$:

PROPOSITION 5.3.1. *The solutions V_ρ of the problems P_ρ verify*

(a) $\lim_{\rho \to +\infty} D(V_\rho) = 0.$

(b) $\lim_{\rho \to 0} R(V_\rho) = 0.$

(c) $D(V_\rho) \leqslant D(V_0), \qquad \forall \rho \in [0, +\infty].$

(d) $R(V_\rho) \leqslant R(V_{+\infty}), \qquad \forall \rho \in [0, +\infty].$

(e) $$\|V_\rho\|^2_{\mathscr{X}} = \sum_{i=1}^{3} \|V_\rho(X_i)\|^2 + D(V_\rho) + R(V_\rho)$$

$$\leqslant \sum_{i=1}^{3} \|V_i\|^2 + D(V_0) + R(V_{+\infty}), \qquad \forall \rho \in [0, +\infty].$$

Proof. The optimality of V_ρ implies that $J_\rho(V_\rho) \leqslant \rho D(V_0)$ and $J_\rho(V_\rho) \leqslant R(V_{+\infty})$. The results follow.

For the same reasons, we note that for $\rho, \rho' \in]0, +\infty[$, the inequalities $J_\rho(V_\rho) \leqslant J_\rho(V_{\rho'})$ and $J_{\rho'}(V_{\rho'}) \leqslant J_{\rho'}(V_\rho)$ give $(\rho - \rho')D(V_\rho) \leqslant (\rho - \rho')D(V_{\rho'})$ and $(1/\rho - 1/\rho')R(V_\rho) \leqslant (1/\rho - 1/\rho')R(V_{\rho'})$. Thus $\rho \leqslant \rho' \Rightarrow D(V_\rho) \geqslant D(V_{\rho'})$ and $R(V_\rho) \leqslant R(V_{\rho'})$; i.e., $D(V_\rho)$ is a decreasing function of ρ and $R(V_\rho)$ an increasing function.

THEOREM 5.3.2. $\operatorname{Lim}_{\rho \to 0} V_\rho = V_0$ *and* $\operatorname{Lim}_{\rho \to +\infty} V_\rho = V_{+\infty}$ *in the space* $\mathscr{X}$.

Proof. The proof is given for $\rho \to 0$.

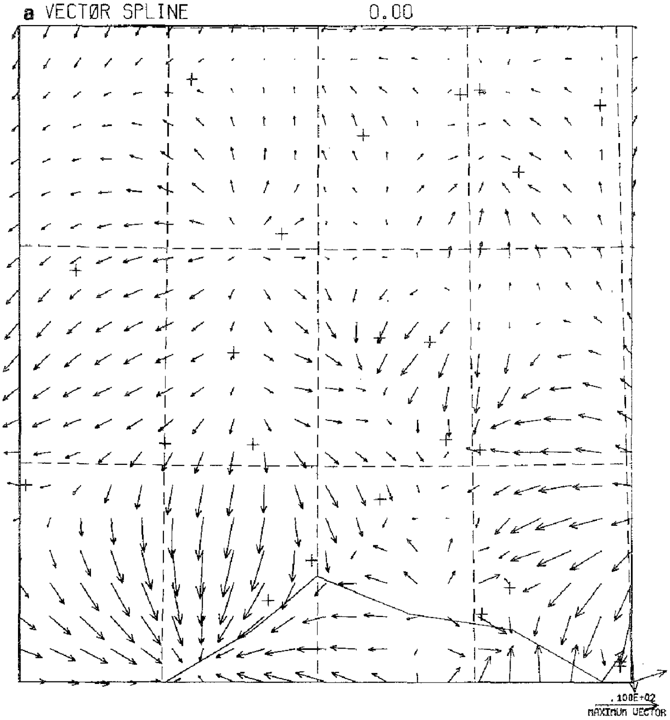
Let (ρ_n) be any sequence such that $\operatorname{Lim}_{n \to +\infty} \rho_n = 0$. From Proposition 5.3.1(e), $\|V_{\rho_n}\|_{\mathscr{X}}$ is bounded. Thus, there is a subsequence of (ρ_n), denoted (ρ_k), such that $V_{\rho_k} \to \widetilde{V_0}$ weakly. The weak convergence implies that the continuous linear forms $u(X_i)$ and $v(X_i)$ verify $u_i = u_{\rho_k}(X_i) \to \widetilde{u_0}(X_i)$ and $v_i = v_{\rho_k}(X_i) \to \widetilde{v_0}(X_i)$. Therefore $\widetilde{V_0}(X_i) = V_i$, $i = 1, ..., N$. But, from Proposition 5.3.1(b) and (c) and continuity of $D(V)$ and $R(V)$, we deduce that $0 \leqslant R(\widetilde{V_0}) \leqslant \operatorname{Lim\,inf} R(V_{\rho_k}) = 0$ and $D(\widetilde{V_0}) \leqslant \operatorname{Lim\,inf} D(V_{\rho_k}) \leqslant D(V_0)$. By uniqueness of the solution of P_0, this implies that $\widetilde{V_0} = V_0$. Using again Proposition 5.3.1(b) and (c), we get $\operatorname{Lim\,sup} \|V_{\rho_k}\|^2_{\mathscr{X}} \leqslant \sum_{i=1}^3 \|V_i\|^2 + D(V_0) = \|V_0\|^2_{\mathscr{X}}$. This inequality and the weak convergence $V_{\rho_k} \to V_0$ implies that $V_{\rho_k} \to V_0$ for the norm $\| \;\|_{\mathscr{X}}$ of $\mathscr{X}$ (cf. Brezis [3]). By the same way, we can show that every convergent subsequence of (V_{ρ_n}) converges necessarily to V_0; consequently $\operatorname{Lim}_{\rho \to 0} V_\rho = V_0$.

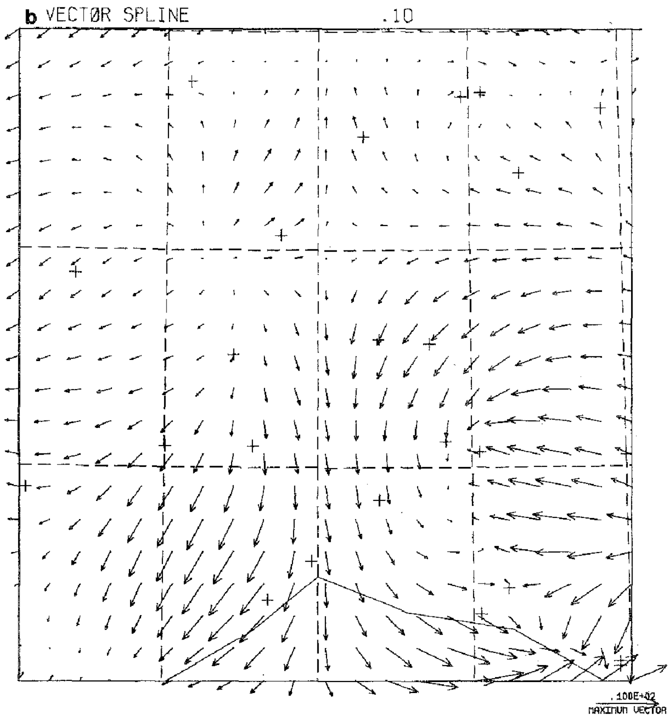
For $\rho \to +\infty$, the proof is similar.

5.4. *Decomposition of the Space* $\mathscr{X}$

The limit problems P_0 and $P_{+\infty}$ have introduced the closed subspaces of $\mathscr{X}$, $\widetilde{\mathscr{X}_{\mathrm{D}}} = \{V \in \mathscr{X} / R(V) = 0\}$ and $\widetilde{\mathscr{X}_{\mathrm{R}}} = \{V \in \mathscr{X} / D(V) = 0\}$. Proposition 2.3 shows that $\widetilde{\mathscr{X}_{\mathrm{D}}} \perp \widetilde{\mathscr{X}_{\mathrm{R}}}$ and $\widetilde{\mathscr{X}_{\mathrm{D}}} \cap \widetilde{\mathscr{X}_{\mathrm{R}}} = \{0\}$ in the subspace $\mathscr{X}_x = \{V \in \mathscr{X} / V(X_i) = 0,\ i = 1, 2, 3\}$. We define the subspaces $\mathscr{X}_{\mathrm{D}} = \{V \in \mathscr{X} / V(X_i) = 0,\ i = 1, 2, 3$ and $R(V) = 0\}$ and $\mathscr{X}_{\mathrm{R}} = \{V \in \mathscr{X} / V(X_i) = 0,$ $i = 1, 2, 3$ and $D(V) = 0\}$.

FIG. 1. Wind fields for $\alpha = 0$ (a), $\alpha = 0.1$ (b), $\alpha = 0.9$ (c), and $\alpha = 1$ (d). The winds are shown by arrows; the scale magnitude is indicated in the lower right-hand corner: the arrow represent a maginitude of $10\ \mathrm{m}\ s^{-1}$.


a VECTØR SPLINE
0.00
.100E+02
MAXIMUM VECTOR


b VECTØR SPLINE
.10
.100E+02
MAXIMUM VECTOR

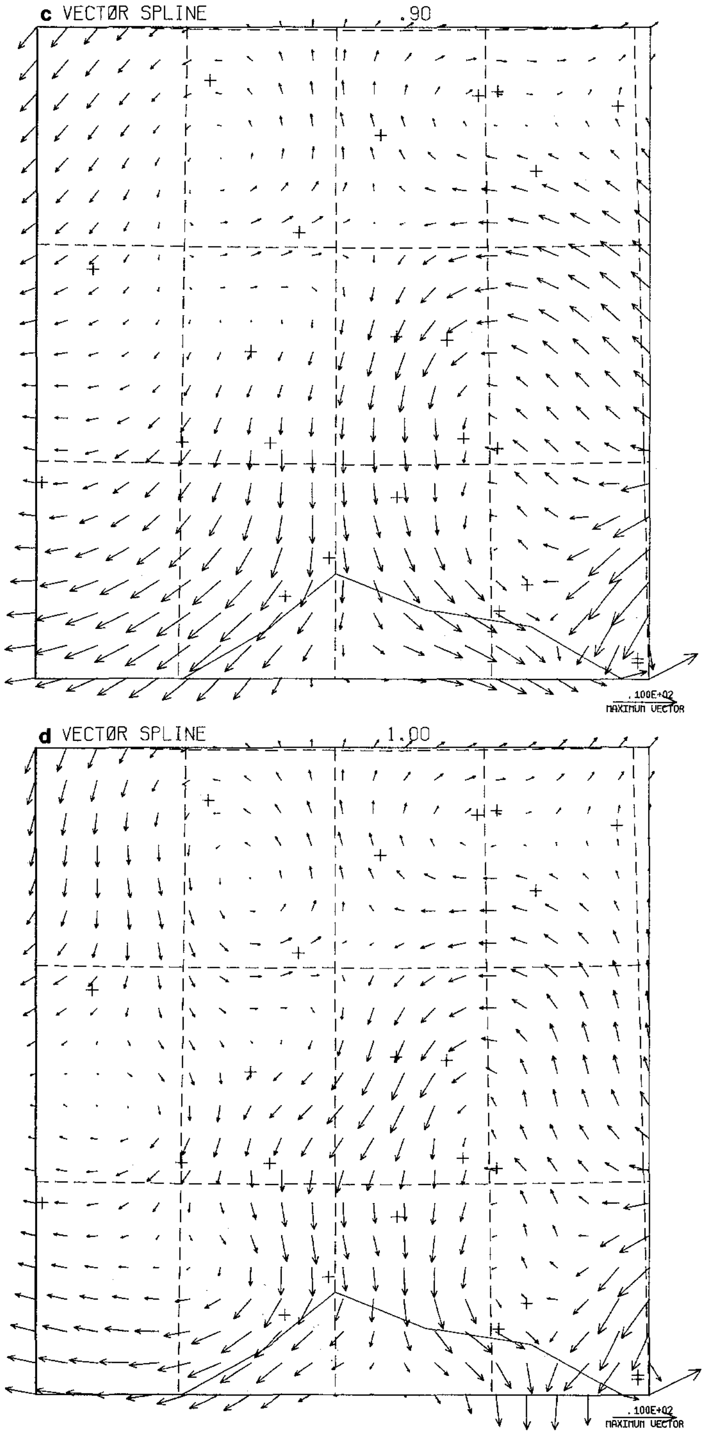


FIG. 1—*Continued*

PROPOSITION 5.4.1. *The Hilbert space* $\mathscr{X}$ *is the direct orthogonal sum of* $(\mathscr{P}_1)^2$, $\mathscr{X}_{\mathrm{D}}$, *and* $\mathscr{X}_{\mathrm{R}}$.

Proof. It is easy to see that $\mathscr{H}$ is the direct orthogonal sum of $(\mathscr{P}_1)^2$ and $\mathscr{H}_X$ (cf. Benbourhim [2]) and that $\mathscr{H}_{\mathrm{D}} \perp \mathscr{H}_{\mathrm{R}}$, $\mathscr{H}_{\mathrm{D}} \cap \mathscr{H}_{\mathrm{R}} = \{0\}$ (Proposition 2.3). We show now that $\mathscr{H}_X = \mathscr{H}_{\mathrm{D}} \oplus \mathscr{H}_{\mathrm{R}}$, i.e., $\mathscr{H}_{\mathrm{D}}^{\perp} = \mathscr{H}_{\mathrm{R}}$ in the space $\mathscr{H}_X$. Let $V \in \mathscr{H}_X$ such that $\langle V \mid V' \rangle = 0$, $\forall V' \in \mathscr{H}_{\mathrm{D}}$. We choose $V' \in \mathscr{H}_{\mathrm{D}}$ of the form $\nabla\varphi + (p, q)$, $\varphi \in \mathscr{D}$ and $p, q \in \mathscr{P}_1$ such that $p(X_i) = -\partial_x \varphi(X_i)$ and $q(X_i) = -\partial_y \varphi(X_i)$, $i = 1, 2, 3$, so that $V'(X_i) = 0$ and $V' \in \mathscr{H}_X$. We derive the equality $\int_{\mathbb{R}^2} (\nabla \operatorname{div} V \cdot \nabla \operatorname{div} V')\, dx\, dy = \int_{\mathbb{R}^2} (\nabla \operatorname{div} V \cdot \nabla \Delta\varphi)\, dx\, dy = \int_{\mathbb{R}^2} (\Delta^2 \operatorname{div} V)\varphi\, dx\, dy = 0$, which shows that $\Delta^2 \operatorname{div} V = 0$ since φ is any element of $\mathscr{D}$. With the same arguments as in Proposition 2.1, we deduce that div V is a constant and therefore $D(V) = 0$. This concludes the proof.

This space decomposition shows that $J_{\alpha,\beta}(V) = \alpha \|P_{\mathrm{D}}(V)\|_{\mathscr{X}}^2 + \beta \|P_{\mathrm{R}}(V)\|_{\mathscr{X}}^2$, where $P_{\mathrm{D}}: \mathscr{H} \to \mathscr{H}_{\mathrm{D}}$ and $P_{\mathrm{R}}: \mathscr{H} \to \mathscr{H}_{\mathrm{R}}$ are the projection operators. Hence, the functional is the norm of a weighted sum of the $\mathscr{H}_{\mathrm{D}}$ and $\mathscr{H}_{\mathrm{R}}$ components of V.

6. NUMERICAL EXAMPLE

We have programmed and tested the method by interpolating real wind data obtained from an operational meteorological observing system. This example is intended to show the variations of the vector field and of the derived divergence and rotational fields along with the parameters α, β (in this case, we consider $\alpha + \beta = 1$). See Figs. 1–3.

The data are 10-m wind measurements (horizontal wind) at 24 station locations, observed the 5th of March 1990, 00 GMT. The considered region is from 43°N–46°N, 2°E–6°E (southeast France). The cartesian coordinates of the station locations are obtained through a stereographic projection; the tangent plane on which the interpolation is done intersects the sphere at 44°30′N, 4°E in the center of the domain. Station locations are noted by a cross (+) and latitude–longitude circles by dashed lines (1° apart). The solid line near the bottom side of the pictures represent the coastal line. In the upper right-hand corner is indicated the value of the parameter α.

For $\alpha = 0$ (P_0 problem), the variations of the divergence field are maximum, while the rotational field is constant ($\simeq 1.3\ 10^{-5} s^{-1}$). The converse happens for $\alpha = 1$ ($\beta = 0$, $P_{+\infty}$ problem): the divergence field is constant ($\simeq 1.7\ 10^{-5} s^{-1}$) and the variations of the divergence field are maximum. However, this example, as a result of the use of ground data, does not present a marked tendency for rotational or divergent behaviour.

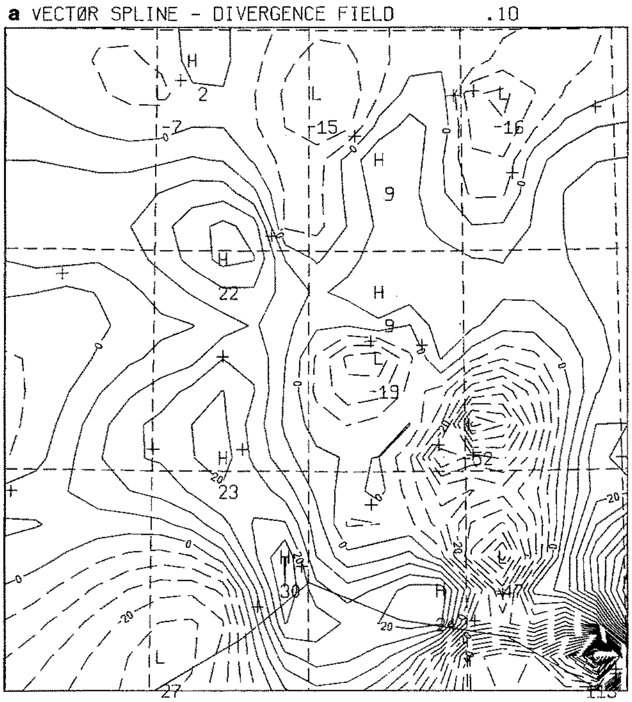


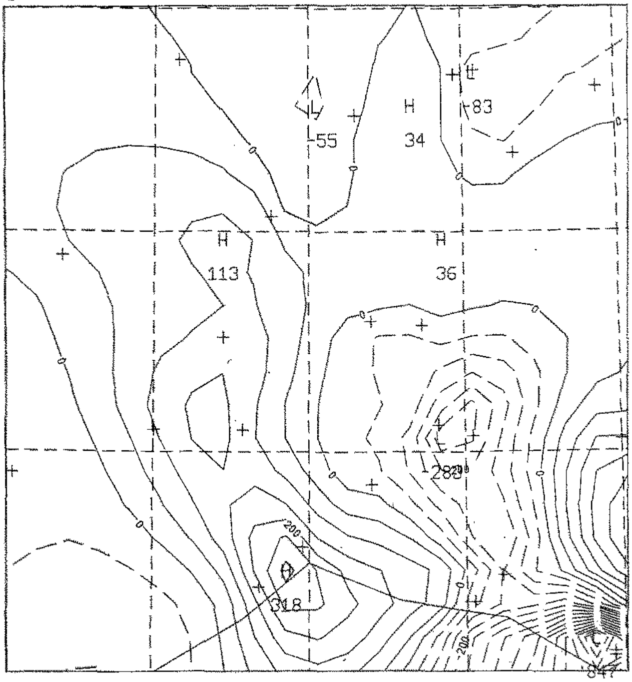
FIG. 2. Divergence field derived from the wind field for $\alpha = 0.1$ (2a) corresponding to Fig. 1(b), $\alpha = 0.5$ (b); $\alpha = \beta = 0.5$; uncoupled wind components); and $\alpha = 0.9$ (c), corresponding to Fig. 1(c). Contours are spaced at $5 \times 10^{-5} s^{-1}$ and are labeled in units of $10^{-5} s^{-1}$ in (a) and $10^{-6} s^{-1}$ in (b) and (c). Negative contours are indicated by dashed lines.

7. Concluding Comments

7.1. *Optimal Values of the Parameters α and β*

For a fixed set of data values, an "optimal" value of the parameter ρ ($\rho = \alpha/\beta$) has to be selected. In the event that an a priori estimate of the ratio $D(V)/R(V)$ (cf. Introduction, Section 1) is available, we obtain the solution by solving iteratively the equation $D(V_\rho)/R(V_\rho) = \rho'$ ($\rho' > 0$ fixed): since $D(V_\rho)/R(V_\rho)$ is a continuous decreasing function of ρ and $\text{Lim}_{\rho \to 0}(D(V_\rho)/R(V_\rho)) = +\infty$, $\text{Lim}_{\rho \to +\infty}(D(V_\rho)/R(V_\rho)) = 0$, we get a unique solution for each fixed value $\rho' > 0$.

However, in the general case, a more intrinsic approach for the determination of this parameter is desirable. Without going deeper into the b VECTØR SPLINE - DIVERGENCE FIELD .50



c VECTØR SPLINE - DIVERGENCE FIELD .90

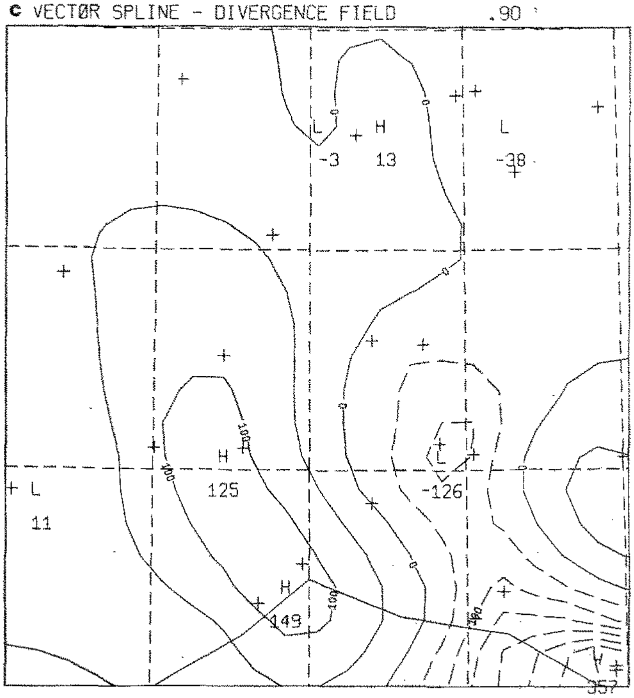


FIG. 2—*Continued*

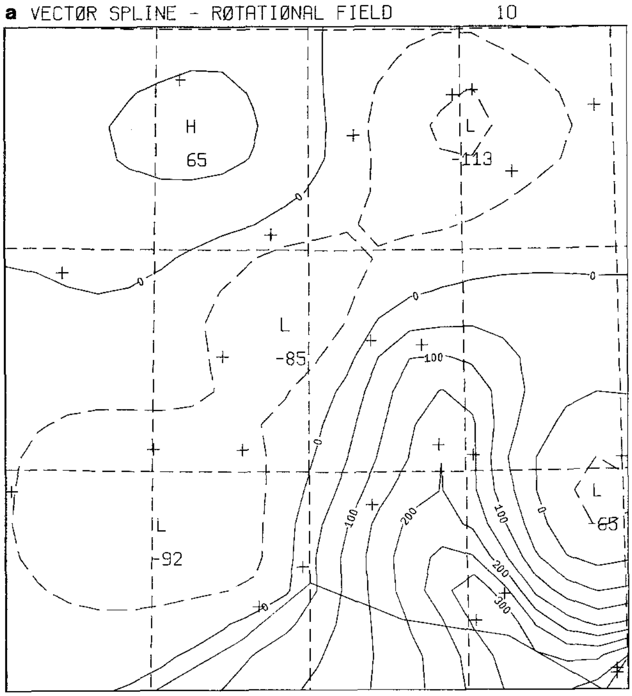


FIG. 3. Rotational field derived from the wind field for the same values of α as in Fig. 2.: $\alpha=0.1$ (a), $\alpha=0.5$ (b), and $\alpha=0.9$ (c). Contours are labeled in units of $10^{-6}s^{-1}$ in (a) and (b), and $10^{-5}s^{-1}$ in (c). Other details as in Fig. 2.

discussion of this problem, we can briefly show how the parameters α and β can be interpreted in a statistical sense. This is the purpose of the next sub-section. Another approach is given by Generalized Cross Validation (GCV) (cf. Wahba and Wendelberger [18] and Utreras [17]), originally introduced for the estimate of smoothing parameters. Some numerical experiments, carried out from analytic data, tend to show its ability to determine the parameter ρ.

7.2. *Relation to the Statistical Interpolation Procedure (Minimum Variance Linear Estimate)*

We first recall the theory for a scalar field $\Phi(X)$, $X\in\mathbb{R}^2$.

We suppose that $\Phi(X)$ is a stochastic process with zero mean value ($E[\Phi(X)]=0$, $\forall X\in\mathbb{R}^2$) and with covariance function $c(X, Y)=E[\Phi(X)\Phi(Y)]$, $X, Y\in\mathbb{R}^2$; Φ is then called a stationary process.

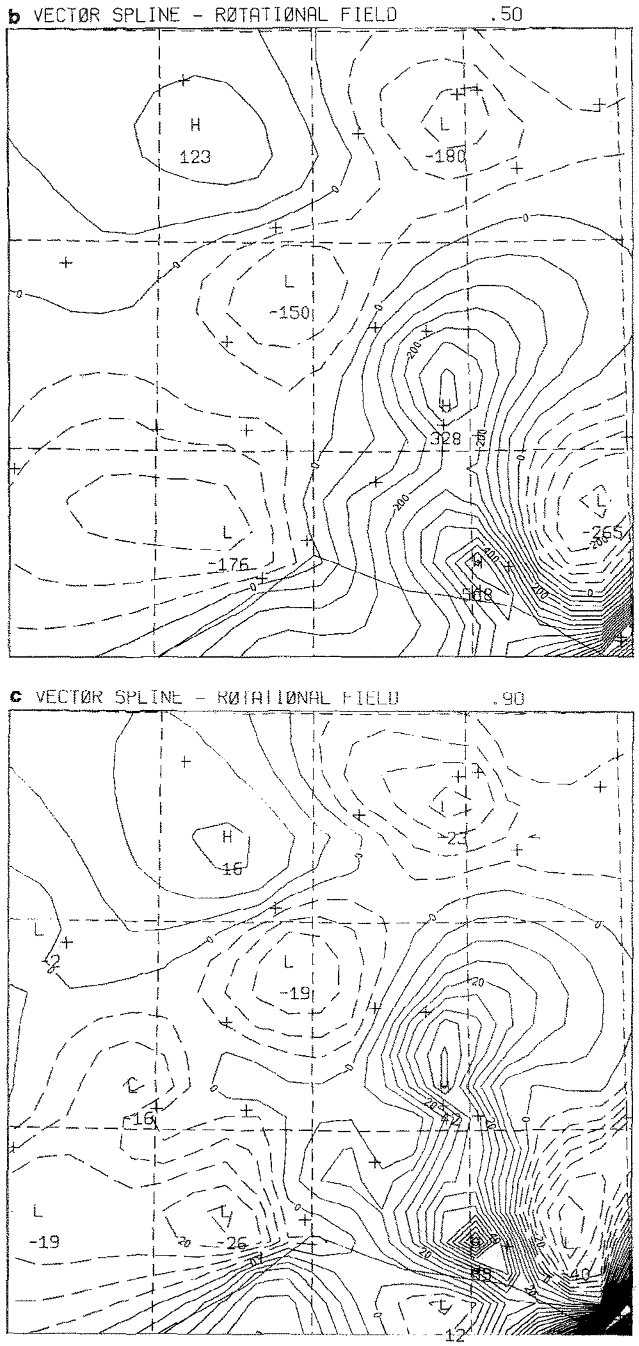


FIG. 3—*Continued*

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For a fixed $X \in \mathbb{R}^2$, we consider the minimum variance linear estimate $\Phi_e(X)$ of $\Phi(X)$, from given values $\Phi(X_i)$, $i=1, ..., N$ (the data). $\Phi_e(X) = \sum_{i=1}^N \lambda_i \Phi(X_i)$, and $\lambda_i \in \mathbb{R}$ are obtained from the minimization problem

$$\operatorname*{Min}_{\lambda} E\left[(\Phi(X) - \sum_{i=1}^{N} \lambda_i \Phi(X_i))^2 \right];$$

λ_i are given by

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix} = C^{-1} \begin{pmatrix} c(X, X_1) \\ \vdots \\ c(X, X_N) \end{pmatrix},$$

where C is the $N \times N$ covariance matrix, $C = (c(X_i, X_j))$.

For a vector field $V(X) = (u(X), v(X))$, the minimum variance linear estimate $V_e(X)$ of $V(X)$ from given values $V(X_i) = (u(X_i), v(X_i))$, $i=1, ..., N$, is

$$u_e(X) = \sum_{i=1}^{N} \lambda_i u(X_i) + \sum_{i=1}^{N} \mu_i v(X_i) \qquad \text{and}$$

$$v_e(X) = \sum_{i=1}^{N} \lambda'_i u(X_i) + \sum_{i=1}^{N} \mu'_i v(X_i),$$

with λ, μ such that $\operatorname{Min}_{\lambda,\mu} E[(u(X) - \sum_{i=1}^N \lambda_i u(X_i) - \sum_{i=1}^N \mu_i v(X_i))^2]$ and λ', μ' such that $\operatorname{Min}_{\lambda',\mu'} E[(v(X) - \sum_{i=1}^N \lambda'_i u(X_i) - \sum_{i=1}^N \mu'_i v(X_i))^2]$. λ and μ are given by

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \\ \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = \begin{pmatrix} C_{uu} & C_{uv} \\ C_{vu} & C_{vv} \end{pmatrix}^{-1} \begin{pmatrix} c_{uu}(X, X_1) \\ \vdots \\ c_{uu}(X, X_N) \\ c_{uv}(X, X_1) \\ \vdots \\ c_{uv}(X, X_N) \end{pmatrix}$$

and λ', μ' are given by

$$\begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_N \\ \mu'_1 \\ \vdots \\ \mu'_N \end{pmatrix} = \begin{pmatrix} C_{uu} & C_{uv} \\ C_{vu} & C_{vv} \end{pmatrix}^{-1} \begin{pmatrix} c_{vu}(X, X_1) \\ \vdots \\ c_{vu}(X, X_N) \\ c_{vv}(X, X_1) \\ \vdots \\ c_{vv}(X, X_N) \end{pmatrix}$$

where $C_{uu} = (c_{uu}(X_i, X_j))$, $C_{uv} = (c_{uv}(X_i, X_j))$, $C_{vu} = C^t_{uv}$, and $C_{vv} = (c_{vv}(X_i, X_j))$ are $N \times N$ covariance matrices ($c_{uu}(X, Y) = E[u(X)u(Y)]$, $c_{uv}(X, Y) = E[u(X)v(Y)]$, $c_{vu}(X, Y) = E[v(X)u(Y)]$, and $c_{vv}(X, Y) = E[v(X)v(Y)]$).

In order to control the divergent and rotational part of the vector field, these covariances are generally deduced from the covariances of χ and ψ (we recall the equality $V = \nabla\chi + \text{Rot}\,\psi$; cf. Section 2, Remark 4).

If now χ, ψ are zero-mean stationary stochastic processes, independent ($E[\chi(X)\psi(Y)] = 0$, $\forall X, Y \in \mathbb{R}^2$), and with covariance functions defined by

$$\begin{aligned} E[\chi(X)\chi(Y)] &= \lambda_\chi c(X - Y) \\ E[\psi(X)\psi(Y)] &= \lambda_\psi c(X - Y), \end{aligned} \tag{7.1}$$

$\lambda_\chi, \lambda_\psi \in \mathbb{R}$, $c(X) = c(-X)$, and if χ, ψ are supposed sufficiently regular, we can derive the following relations for (u, v):

$$\begin{aligned} E[u(X)u(Y)] &= \lambda_\chi \partial^2_{xx} c(X - Y) + \lambda_\psi \partial^2_{yy} c(X - Y) \\ E[v(X)v(Y)] &= \lambda_\psi \partial^2_{xx} c(X - Y) + \lambda_\chi \partial^2_{yy} c(X - Y) \\ E[u(X)v(Y)] &= E[u(Y)v(X)] = (\lambda_\chi - \lambda_\psi)\partial^2_{xy} c(X - Y). \end{aligned} \tag{7.2}$$

Formally, we can identify the vector spline $V_{\alpha,\beta}$ with the minimum variance linear estimate obtained with u and v defined by the covariances (7.2) and with $\lambda_\chi = 1/\alpha$, $\lambda_\psi = 1/\beta$, and $c(X) = K(X)$. In fact, the identification may be stated rigorously if the stochastic processes χ, ψ are not considered stationary but, following the terminology of Kriging (cf. Matheron [13], Dubrule [6, 7]), supposed to have stationary increments of order 2 and variograms of the form (7.1). The deduced u, v processes have stationary increments of order 1 and variograms given by (7.2). The details of this point will be developed in a future work.

7.3. Extensions

7.3.1. *Smoothing Spline*

In the same way as for all spline functions, we can introduce a smoothing problem: For fixed λ, α, $\beta > 0$, we define

$$\begin{cases} \text{Min}(\lambda J_{\alpha,\beta}(V) + \sum_{i=1}^{N} \|V(X_i) - V_i\|^2 \\ V \in \mathscr{X}. \end{cases}$$

From the above theory, the unique solution of this problem is easily deduced.

7.3.2. *Three Dimensional Vector Spline*

A three dimensional vector interpolation using div and rot operators (rot is now a vector operator) can be formulated in the same way. Denoting $V(X)=(u(X), v(X), w(X))$, $X=(x, y, z)$, $\operatorname{div} V=\partial_x u+\partial_y v+\partial_z w$, and

$$\operatorname{rot} V=\begin{pmatrix}(\operatorname{rot} V)_x\\(\operatorname{rot} V)_y\\(\operatorname{rot} V)_z\end{pmatrix}=\begin{pmatrix}\partial_y w-\partial_z v\\\partial_z u-\partial_x w\\\partial_x v-\partial_y u\end{pmatrix},$$

the functional takes the form

$$\begin{aligned}J_{\alpha,\beta_x,\beta_y,\beta_z}(V)=\alpha&\int_{\mathbb{R}^3}\|\nabla \operatorname{div} V\|^2\,dx\,dy\,dz\\&+\beta_x\int_{\mathbb{R}^3}\|\nabla(\operatorname{rot} V)_x\|^2\,dx\,dy\,dz\\&+\beta_y\int_{\mathbb{R}^3}\|\nabla(\operatorname{rot} V)_y\|^2\,dx\,dy\,dz\\&+\beta_z\int_{\mathbb{R}^3}\|\nabla(\operatorname{rot} V)_z\|^2\,dx\,dy\,dz,\end{aligned}$$

where ∇ is the three dimensional nabla operator, $\|\ \|$ the usual euclidean norm of $\mathbb{R}^3$, and α, β_x, β_y, β_z positive constants. For $\alpha=\beta_x=\beta_y=\beta_z$, we get the same result as in Proposition 2.3, $J_{\alpha,\alpha,\alpha,\alpha}(V)=\alpha[((u\,|\,u))+((v\,|\,v))+((w\,|\,w))]$, and in this case the interpolation is applied separately to the three components (here $((\ |\ ))$ is the three dimensional version of the semi-scalar product introduced in Section 2).

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References

1. M. Atteia and M. N. Benbourhim, Spline elastic manifolds, *in* "Mathematical Methods in Computer Aided Geometric Design" (T. Lyche and L. L. Schumaker, Eds.), pp. 45–50, Academic Press, Boston, 1989.
2. M. N. Benbourhim, "Fonctions splines d'approximation," Thèse de troisième cycle, Toulouse, 1982.

3. H. Brezis, "Analyse fonctionnelle, théorie et applications," Masson, Paris, 1983.
4. R. Daley, "Spectral Characteristics of the ECMWF Objective Analysis Scheme," Technical Note No. 40, ECMWF, Reading, U.K., 1983.
5. R. Daley, The analysis of synoptic-scale divergence by a statistical interpolation scheme, *Monthly Weather Rev.* **113** (1985), 1066–1079.
6. O. Dubrule, "Krigeage et splines en cartographie automatique," Thèse de Docteur-Ingénieur, Ecole Nat. Sup. des Mines de Paris, 1981.
7. O. Dubrule, Cross validation of kriging in a unique neighborhood, *Math. Geol.* **15** (6) (1983).
8. J. Duchon, Interpolation des fonctions de deux variables suivant le principe de la flexion des plaques minces, *RAIRO Anal. Numér.* **10** (12) (1976).
9. A. Eliassen, "Provisional Report on Calculation of Spatial Covariance and Autocorrelation of the Pressure Field," Report No. 5, Inst. Weather and Climate Res., Acad. Sci. Oslo, 1954.
10. L. Gandin, "Objective Analysis of Meteorological Fields," Israel Program for Scientific Translation, Jerusalem, 1963.
11. G. J. Haltiner and R. T. Williams, "Numerical Prediction and Dynamic Meteorology," Wiley, New York, 1980.
12. P. J. Laurent, "Approximation et optimisation," Hermann, Paris, 1972.
13. G. Matheron, The intrinsic random functions and their applications, *Adv. in Appl. Probab.* No. 5 (1973), 439–468.
14. J. Necas, "Les méthodes directes en théorie des équations elliptiques," Masson, Paris, 1967.
15. M. A. Pedder, On the influence of map analysis formulation in the estimation of wind field derivatives, *Quart. J. Roy. Meteorol. Soc.* **114** (1988), 241–257.
16. L. Schwartz, "Théorie des distributions," Hermann, Paris, 1978.
17. F. Utreras, "Utilisation de la méthode de validation croisée pour le lissage par fonction spline à une ou deux variables," Thèse de Docteur-Ingénieur, Grenoble, 1979.
18. G. Wahba and J. Wendelberger, Some new mathematical methods for variational objective analysis using splines and cross-validation, *Monthly Weather Rev.* **108** (1980), 1122–1143.